Notes on Chern's Affine Bernstein Conjecture

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Abstract. There were two famous conjectures on *complete affine maximal surfaces*, one due to E. Calabi, the other to S.S. Chern. Both were solved with different methods about one decade ago by studying the associated Euler-Lagrange equation. Here we survey two proofs of Chern's conjecture in our recent monograph [L-X-S-J], in particular we add some details of the proofs of auxiliary material that were omitted in [L-X-S-J]. We describe the related background in our Introduction. Our survey is suitable as a report about recent developments and techniques in the study of certain Monge-Ampère equations.

1. Introduction

Many geometric problems in analytic formulation lead to important classes of PDEs. Naturally, geometric methods play a crucial role in their investigation. Typical examples are the *Euclidean Minkowski Problem*, the Theorem of Jörgens-Calabi-Pogorelov on the *global uniqueness of improper affine spheres*, or the two famous *Affine Bernstein Conjectures* of Calabi and Chern, resp.

Wide use of geometric methods in studying PDE's of affine hypersurface theory was initiated by E. Calabi and continued by A.V. Pogorelov, S.Y. Cheng - S.T. Yau, A.-M. Li, and, during the last decade, e.g. by N.S. Trudinger - X.J. Wang, A.-M. Li's school, and other authors. An effective application of such geometric methods has played a crucial role in these studies.

The contributions of E. Calabi and S.Y. Cheng - S.T. Yau had a particularly deep influence on the development of this subject. According to the foreword in [Ch-Y-86] this paper originated from discussions with E. Calabi and L. Nirenberg,

and from results of both on the same topic; for further historical details and references we refer to [Ca], [CA], [POG-72], [POG-75] and the monographs [L-S-Z], [L-X-S-J].

In problems involving PDE's of Monge-Ampère type it is often the case that the unknown solution is a convex function defining locally a nonparametric hypersurface for which it is possible to choose a suitable relative normalization and investigate the induced geometry. We refer to this process as *geometric modelling*, described and emphasized in the introduction of the monograph [L-X-S-J].

Our emphasis is different here. Even when there is some parallelity in tools, that is Lemmas and Propositions, used in [L-X-S-J] and also here, here our emphasis is on the following:

In recent years, A.-M. Li and his school extended the study of certain types of Monge-Ampère equations and developed a framework, in particular for studying such types of 4-th order PDE's, including the affine maximal hypersurface equation and the Abreu equation (cf. [L-J-1], [L-J-2], [L-J-3], [L-J-4], [L-X-1],[L-X-2], [C-L-S-2], [C-L-S-4]).

We call this the *real affine technique*. The whole package includes:

- the derivation of differential inequalities for certain functions of geometric importance, related to the given problem; here that is the differential inequality (4.3) for the function Φ;
- convergence theorems,
- Bernstein properties and
- an affine blow-up analysis.

This package is very useful for studying such types of PDEs; the technique was extended to complex manifolds; it also plays an important role in the study of extremal metrics on toric surfaces; see [C-L-S-1] and [C-L-S-3].

It is the central aim of this paper to survey this *real affine technique* and to sketch two proofs of Chern's conjecture in our recent monograph [L-X-S-J], in particular to outline details of the proof of Proposition 5.6.13, which were omitted in [L-X-S-J]; see section 3 below.

Chern's Conjecture.   *Let* $x_{n+1} = f(x_1, x_2, ..., x_n)$ *be a smooth, strictly convex function defined for all* $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. *If the graph hypersurface*

$$M = \{(x, f(x) \mid x \in \mathbb{R}^n\}$$

is affine maximal then M *must be an elliptic paraboloid.*

The two-dimensional conjecture of Chern was solved by N. Trudinger and X. Wang in [T-W-1]; they combined *tools from the theory of convex bodies* and from the *Caffarelli-Gutiérrez theory*. In [L-J-3] one can find a different proof, also using tools from the theory of convex bodies and the Caffarelli-Gutiérrez theory. In [L-J-1] and [L-X-S-J] the authors gave a completely different proof of Chern's conjecture, purely using tools from analysis.

Concerning this purely analytic proof of Chern's conjecture, our proof here also meets demands of other geometers; namely, in [L-X-S-J], p. 126, we omitted the

proof of Proposition 5.6.13 and Proposition 5.6.15, stating that both proofs are very "similar" to the proofs of the foregoing Propositions 5.6.12 and 5.6.14, resp., following exactly the steps of these proofs.[1] We decided to include some more details here as *exemplary demonstration of the real affine techniques*. For the convenience of the reader we give, where needed, precise references to [L-X-S-J] for a parallel reading, and guide the reader to extended results.

Let us give a more precise description of the type of equation we are going to investigate here: We study *Bernstein Properties* of a nonlinear, fourth order partial differential equation for a convex function f on a convex domain $\Omega \subset \mathbb{R}^n$; this equation can be written as

$$\sum_{i,j=1}^{n} F^{ij} w_{ij} = 0, \quad w := \left[\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^a, \tag{1.1}$$

where $a \neq 0$ is a real constant and where (F^{ij}) denotes the cofactor matrix of the Hessian matrix $(f_{ij}) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$.

Denote by Δ and $\| \cdot \|$ the Laplacian and the tensor norm with respect to the *Calabi metric* $G := \sum f_{ij} dx_i dx_j$, respectively. Set

$$\rho := [\det(f_{ij})]^{-\frac{1}{n+2}} \quad \text{and} \quad \Phi := \frac{1}{\rho^2} \cdot \|\operatorname{grad} \rho\|^2.$$

In terms of the Calabi metric the PDE (1.1) can be rewritten as

$$\Delta \rho = -\beta \, \frac{\|\operatorname{grad} \rho\|^2}{\rho}, \tag{1.2}$$

where

$$\beta := -\frac{(n+2)(2a+1)+2}{2}.$$

The PDE (1.1) appears in different geometric problems, namely:

- *Chern's Conjecture:* In case when $a = -\frac{n+1}{n+2}$, the PDE (1.1) is the equation for affine maximal hypersurfaces; this is the equation that is related to Chern's Conjecture cited above; for Chern's Conjecture we have $n = 2$ and $\beta = 0$.
- *Abreu equation:* In case that $a = -1$ the PDE (1.1) appears in the study of the differential geometry of toric varieties (see [A], [D]).

Remark. Chern's completeness assumption is given by the fact that the function f is assumed to be defined for all $x \in \mathbb{R}^n$; this completeness assumption is affinely invariant, in affine hypersurface theory it is called *Euclidean completeness.* Instead Calabi assumed that the *Blaschke metric* of the affine maximal surface should be complete; this completeness assumption is called *affine completeness.*

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[1]X. Wang from the Australian National University, another expert in the field, wrote a personal letter to the third author; he complained that our monograph [L-X-S-J] does not contain the proofs of the auxiliary Proposition 5.6.13 and Proposition 5.6.15, resp.

About the Bernstein property for the Abreu equation, A.-M. Li and F. Jia posed the following conjecture in [L-J-1]:

Conjecture of Li-Jia. *Let f be a smooth, strictly convex function defined for all $x \in \mathbb{R}^n$. Assume that f satisfies the Abreu equation*

$$\sum_{i,j=1}^{n} F^{ij} w_{ij} = 0, \quad w := \left[\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^{-1}.$$

Then f must be a quadratic polynomial.

In [L-J-1] Li and Jia proved the following theorem:

Theorem 1. (Theorem 5.6.2 in [L-X-S-J]). *Let f be a smooth, strictly convex function defined for all $(x_1, x_2) \in \mathbb{R}^2$. If f satisfies the PDE*

$$\sum_{i,j=1}^{2} F^{ij} w_{ij} = 0, \quad w := \left[\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^{a}$$

with $a \leq -\frac{3}{4}$, then f must be a quadratic polynomial.

Remark.
(1) When $a = -\frac{3}{4}$, Theorem 1 gives a new proof for Chern's conjecture on complete affine maximal surfaces. That is a particular topic of this paper.
(2) When $a = -1$, Theorem 1 affirmatively solves the above Conjecture of Li-Jia for $n = 2$.
(3) In [T-W-2], N. Trudinger and X. Wang proved that the global solution of the PDE (1.1) with $a > 0$ on $\mathbb{R}^2$ must be a quadratic polynomial.
(4) When $a = 0$ and $n = 2$, the above PDE (1.2) reads $\Delta \rho = 3 \frac{\|\operatorname{grad} \rho\|^2}{\rho}$;
this is equivalent to

$$\sum_{i,j=1}^{2} f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} [\ln \det(f_{kl})] = 0. \tag{1.3}$$

(5) The global solution of the PDE (1.3) on $\mathbb{R}^2$ is not unique. In fact, the following are different global solutions of (1.3):

$$f(x_1, x_2) = \sum_{i=1}^{2} x_i^2, \qquad h(x_1, x_2) = e^{x_1} + x_2^2.$$

In our foregoing sketch we indicated tools from analysis and geometry by catchphrases of experts. In the following short list we give some hints and references:

- *Caffarelli-Schauder estimates:*
  [CAF] gives interior a priori estimates for solutions of perturbations of Monge-Ampère equations; [CAF-1] treats a localization property of convex viscosity solutions of certain Monge-Ampère inequalities.

- *Caffarelli-Gutiérrez theory:*
  The paper [CG] treats the properties of the solutions of a linearized Monge-Ampère equation.
- *Affine blow-up analysis:*
  One can apply a series of affine transformations to rescale the domain and the function such that the geometric invariant to be estimated has a better behavior. Such a procedure is called *affine blow-up analysis.* In this process we often use a Lemma of Hofer. Affine blow-up analysis is a very powerful tool to estimate affine invariants in various circumstances, see [L-J-2], [L-J-3], [L-J-4], [C-L-S-3].
- *Tools from the theory of convex bodies :*
  A Theorem of Alexandrov-Pogorelov-Heinz gives sufficient conditions for the local convexity of a surface; such conditions are of importance for the solution of certain elliptic Monge-Ampère equations; see [H] and [BU], p. 35.
- *Affine and relative hypersurface theories:*
  The extrinsic curvature functions in the affine theories are of fourth order, in particular the Euler-Lagrange equation for the *Affine Bernstein Conjecture*; the invariants that characterize quadrics play an important role for the solution of certain Monge-Ampère equations; for details see the monographs [L-S-Z] and [S-S-V].

In [L-X-S-J] we presented an analytic proof of Theorem 1. In this note we outline a proof with emphasis on the proof of Chern's conjecture.

2. Proof of Theorem 1.

Let $C > 0$ be given a real constant and a convex domain $\Omega \subset \mathbb{R}^n$; denote by $\mathcal{S}(\Omega, C)$ the class of strictly convex C^∞-functions f, defined on Ω, such that

$$\inf_\Omega f(x) = 0, \quad f(x) = C \quad \text{on} \quad \partial\Omega.$$

We use the notion of a *normalized convex set* in the sense stated on p. 48 in [G]. To prove Theorem 1 we need the following Lemma:

Lemma 2.1. (Lemma 5.6.16 in [L-X-S-J]). *Let $\Omega_k \subset \mathbb{R}^2$ be a sequence of smooth normalized convex domains, converging to a convex domain Ω, and let $f^{(k)} \in \mathcal{S}(\Omega_k, C)$ with $f^k(q^k) = 0$. Assume that the functions $f^{(k)}$ satisfy the PDE (1.2) with $\beta \geq 0$. Then there exists a subsequence $f^{(i_\ell)}$ that locally uniformly converges to a convex function $f \in C^0(\Omega)$ with distance $dist(p_o, \partial\Omega) > 0$, where p_o is the point such that $f(p_o) = 0$. Moreover, there is an open neighborhood N of p_o such that $f^{(i_\ell)}$ converges to f, and also all their derivatives converge, therefore f is smooth and strictly convex in N.*

We will sketch the proof of this Lemma in Section 3.

Proof of Theorem 1. Let $x : M \to \mathbb{R}^3$ be a locally strongly convex surface, given as graph of a smooth, strictly convex function f, defined for all $(x_1, x_2) \in \mathbb{R}^2$. Assume that f satisfies the PDE (1.2) with $\beta \geq 0$. Given any $p \in M$, - adding a linear function if necessary, - we may assume that

$$f(p) = 0, \qquad \frac{\partial f}{\partial x_i}(p) = 0, \quad i = 1, 2.$$

Choose a sequence $\{C_k\}$ of real positive numbers such that $C_k \to \infty$ as $k \to \infty$. Then, for any C_k, the *section*

$$S_f(p, C_k) := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid f(x_1, x_2) < C_k \right\}$$

is a bounded convex domain in $\mathbb{R}^2$. It is well-known (see section 1.8 in [G]) that there exists a unique ellipsoid E_k which attains the minimum volume among all ellipsoids that contain $S_f(p, C_k)$ and that are centered at the center of mass of $S_f(p, C_k)$ such that

$$2^{-\frac{3}{2}} E_k \subset S_f(p, C_k) \subset E_k .$$

Let T_k be an affine transformation such that

$$T_k(E_k) = B_1(0) = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 < 1 \right\} .$$

Define the functions

$$f^{(k)}(x) := \tfrac{1}{C_k} \, f(T_k^{-1} x).$$

Then

$$B_{2^{-\frac{3}{2}}}(0) \subset \Omega_k \subset B_1(0),$$

where

$$\Omega_k = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid f^{(k)}(x_1, x_2) < 1 \right\} .$$

Taking subsequences, we may assume that $\{\Omega_k\}$ converges to a convex domain Ω and $\{f^{(k)}\}$ converges to a convex function f^∞, locally uniformly in Ω. By Lemma 2.1 the function f^∞ is smooth and strictly convex in a neighborhood of $T_k(p) \in \Omega$, for k large enough. It follows that the functions $\Phi^{(k)}(T_k(p))$ are uniformly bounded. Assume that $\Phi(p) \neq 0$; a direct calculation gives

$$\Phi^{(k)}(T_k(p)) = C_k \Phi(p) \to \infty,$$

thus we get a contradiction, and thus $\Phi(p) = 0$. Since p is arbitrary, we have $\Phi = 0$ everywhere on M. It follows that $\det(f_{ij}) = const$. The Theorem of Jörgens ([L-X-S-J], p. 59) implies that f must be a quadratic polynomial. This completes the proof of Theorem 1. ∎

3. Proof of Lemma 2.1.

In this section we sketch two different proofs of Lemma 2.1. The first one uses the theory of convex bodies combined with the theory of Caffarelli-Gutiérrez, while the second one uses pure analysis. Both proofs need the following estimates for determinants. Here and later we separate the cases $\beta > 0$ and $\beta = 0$.

3.1. Estimates for determinants

We restrict to dimension $n = 2$. For $\beta > 0$ we have:

Lemma 3.1. (Lemma 5.6.9 in [L-X-S-J]). *Let $0 < C \in \mathbb{R}$ and Ω be a normalized convex domain with $0 \in \Omega$ as center of Ω. Assume that $f \in \mathcal{S}(\Omega, C)$ and f satisfies the PDE (1.2) with $\beta > 0$. Let u be the Legendre transformation function of f relative to 0. Then the following estimate holds:*

$$\frac{1}{(d+u)^4} \cdot \det(f_{ij}) \leq b_0 \qquad for \quad x \in S_f(C') = \{x \in \Omega \mid f(x) \leq C'\},$$

where $0 < C' < C$, and b_0 is a constant depending only on $\frac{C'}{C}$, C, and β.

The affine maximal surface equation is affinely invariant; thus, for $\beta = 0$, we can choose a coordinate system x_1, x_2, x_3 such that the graph function $x_3 = f(x_1, x_2)$ satisfies $\|grad\, f\|_E \leq C_0$ for some constant C_0 in a section; here and later $\| \cdot \|_E$ denotes the norm with respect to the canonical Euclidean metric of $\mathbb{R}^2$. From Lemma 5.6.4 in [L-X-S-J] we have

Lemma 3.2. *Let C and Ω be defined as before. Assume that $f \in \mathcal{S}(\Omega, C)$ and f satisfies the PDE (1.2) with $\beta = 0$. If*

$$\|grad\, f\|_E \leq C_0$$

for some constant C_0, then the following estimate holds:

$$\det(f_{ij}) \leq b_0 \qquad for \quad x \in S_f(C'),$$

where b_0 is a constant depending only on $0 < C$, $0 < \frac{C'}{C} < 1$ and C_0.

Lemma 3.3. (Lemma 5.6.6 in [L-X-S-J]). *Let f be a smooth, strictly convex function defined on a bounded convex domain $\Omega \subset \mathbb{R}^2$, satisfying the PDE (1.2). Denote by Ω^* the Legendre transformation domain of f. Let Ω'^* be an arbitrary subdomain of Ω^* with $dist\,(\Omega'^*, \partial\Omega^*) > 0$. Then the following estimate holds:*

$$\det(f_{ij}) \geq b_0 \quad for \quad \xi \in \Omega'^*,$$

where b_0 is a constant depending only on $dist(\Omega'^*, \partial\Omega^*)$, $diam(\Omega)$, $diam(\Omega^*)$ and $\beta \in \mathbb{R}$.

One can use the Lemmata 3.1 - 3.3, the theory of convex bodies, and finally the Caffarelli-Gutiérrez theory to prove Lemma 2.1. This was carried out in [L-J-3] for $\beta = 0$ (see also section 6.1 in [L-X-S-J]). In a remark in [L-J-1], Li and Jia have pointed out that this method works for $\beta > 0$ (see also p. 129 in [L-X-S-J]); but they did not give details. Here we give a proof.

First Proof of Lemma 2.1.

This proof uses the Caffarelli-Gutiérrez theory and tools from the theory of convex bodies.

Case $\beta > 0$. First we consider the case $\beta > 0$. From Lemma 3.1, for the function $f^{(k)}$ in Lemma 2.1, we have the following estimate in the section $S_{f^{(k)}}(q^k, \frac{C}{2})$:

$$\frac{\det(f^{(k)}_{ij})}{(d + u^{(k)})^4} \leq d_0. \tag{3.1}$$

Choose the radius R for a Euclidean ball such that

$$B_R(q^k) \supset \Omega^{(k)} \supset S_{f^{(k)}}(q^k, \tfrac{C}{2}); \qquad \text{then the relation} \qquad B^*_r(0) \subset (\Omega^{(k)})^*$$

is satisfied for the Legendre transformation domains, where $r = \frac{C}{4R}$. Moreover, restricting to $B^*_r(0)$, we have the inequality

$$-\frac{C}{2R} - C \leq u^{(k)} \leq \frac{C}{2R}.$$

Therefore,

1. $u^{(k)}$ locally converges to a convex function u^∞ on $B^*_r(0)$;
2. from standard relations of the Legendre transformation (see section 1.4 in [L-X-S-J]) and from (3.1) we know that $\det(u^{(k)}_{ij}) = [\det(f^{(k)}_{ij})]^{-1} > d_1$ is bounded from below on $B^*_r(0)$.

Then the Alexandrov-Pogorelov-Heinz Theorem implies that u^∞ is strictly convex in a small neighborhood of $(0,0) \in B^*_r(0)$. Adding a linear function if necessary, we may assume that

$$u^\infty(0) = 0, \qquad \frac{\partial u^\infty}{\partial \xi_i}(0) = 0, \quad i = 1, 2.$$

Thus there exists a real constant $h_0 > 0$ such that the section

$$S_{u^\infty}(h_0) := \{\xi \in B^*_{\frac{r}{2}}(0) \mid u^\infty \leq h_0\}$$

is compact in $B^*_{\frac{r}{2}}(0)$. Then, in this section $S_{u^\infty}(h_0)$, we have a lower bound for $\det(u^{(k)}_{ij})$ and, by Lemma 3.3, we also have an upper bound. Now we use the Caffarelli-Gutiérrez theory and the Caffarelli-Schauder estimate to conclude that $\{u^{(k)}\}$ smoothly converges to u^∞ in $\{\xi \mid u(\xi) \leq h_0\}$. Therefore, u^∞ is a smooth and strictly convex function in an open neighborhood of $(0, 0)$ in $B^*_{\frac{r}{2}}(0)$.

Case $\beta = 0$. Next we consider the case $\beta = 0$ (see [L-X-S-J], p.129 and p. 153-159). Denote $D := \{x \mid f(x) = 0\}$. Again we discuss two subcases.

(i) If $D \bigcap \partial\Omega = \emptyset$ then there is a constant $h > 0$ such that the level set satisfies $\bar{S}_f(p_o, h) \subset \Omega$, and thus we have a uniform estimate for $\|grad\, f^{(k)}\|_E$ in $\bar{S}_{f^{(k)}}(q^k, h)$. From Lemma 3.2 it follows that there is a uniform estimate for $\frac{1}{\rho}$ in $\bar{S}_{f^{(k)}}(q^k, \frac{h}{2})$. Then we use the same argument as in the case $\beta > 0$ to conclude that $\{u^{(k)}\}$ smoothly converges to u^∞ in $\{\xi \mid u(\xi) \leq h_0\}$. Therefore u^∞ is a smooth and strictly convex function in an open neighborhood of $(0, 0)$ in $B^*_{\frac{r}{2}}(0)$.

(ii) If $D \bigcap \partial\Omega \neq \emptyset$, let $p \in D \bigcap \partial\Omega$. Since the PDE (1.2) with $\beta = 0$ is equiaffinely invariant, we may choose a new coordinate system such that the term $\|grad\, f^{(k)}\|_E$

is uniformly bounded in $\bar{S}_{f^{(k)}}(p,h)$. Then the same argument shows that f is smooth in a neighborhood of p, and we get a contradiction. This excludes the case $D \bigcap \partial\Omega \neq \emptyset$.

Lemma 2.1 is proved.   ∎

3.2 Estimates for the third derivatives and for $\sum f_{ii}$.

It is interesting to give a purely analytic proof of Lemma 2.1, without using tools from the theory of convex bodies. Here we apply the *real affine technique* that we mentioned in the Introduction. The key point of the proof of Lemma 2.1 are the estimates for the third order derivatives and for $\sum f_{ii}$. We introduce the following notations:

$$\mathcal{A} := \max_{\Omega} \left\{ \exp\left\{ -\frac{m}{C-f} \right\} \frac{\Phi}{\rho^{\alpha}(d+u)^{\alpha}} \right\},$$

$$\mathcal{D} := \max_{\Omega} \left\{ \exp\left\{ -\frac{m}{C-f} + K \right\} \frac{g^2 \, \| \, grad \, f \|^2}{\rho^{\alpha}(d+u)^{\alpha}} \right\},$$

where

$$K := \frac{N}{\mathcal{A}} \exp\left\{ -\frac{m}{C-f} \right\} \frac{\Phi}{\rho^{\alpha}(d+u)^{\alpha}},$$

and m, α and N are positive real constants. As before we separate the cases $\beta > 0$ and $\beta = 0$.

Lemma 3.4. (Lemma 5.6.10 in [L-X-S-J]). *Assume that $\Omega \subset \mathbb{R}^2$ is a normalized domain and $f \in \mathcal{S}(\Omega, C)$ satisfies the PDE (1.2) with $\beta > 0$, and that there exists a constant $b > 0$ such that in Ω:*

$$\frac{1}{\rho(d+u)} < b.$$

Then there are constants $\alpha > 0$, N and m such that

$$\mathcal{A} \leq \max\{d_1, \tfrac{4}{\alpha\beta}\mathcal{D}\},$$

where d_1 is a real constant depending only on α, C, b and β.

Lemma 3.5. (Lemma 5.6.11 in [L-X-S-J]). *Let $\Omega \subset \mathbb{R}^2$ be a normalized convex domain and $f \in \mathcal{S}(\Omega, C)$, satisfying the PDE (1.2) with $\beta > 0$. Assume that there exists a constant $b > 0$ such that in Ω:*

$$\frac{1}{\rho(d+u)} < b.$$

Then there exist real constants $\alpha > 0$, N and m such that

$$\mathcal{A} \leq d_2, \qquad \mathcal{D} \leq d_2$$

for some real constant $d_2 > 0$ that depends only on α, b, β and C.

As a corollary of Lemma 3.4 and Lemma 3.5 we get:

Proposition 3.6. (Proposition 5.6.12 in [L-X-S-J]). *Let $\Omega \subset \mathbb{R}^2$ be a normalized convex domain and $0 \in \Omega$ be the center of Ω. Let f be a strictly convex C^{∞} function*

defined on Ω. Assume that

$$\inf_{\Omega} f = 0, \qquad f = C > 0 \quad on \ \partial\Omega,$$

and that f satisfies the PDE (1.2) with $\beta > 0$. Then there exists a constant $\alpha > 0$ such that, on $\Omega_{\frac{C}{2}} := \{x \in \Omega \mid f(x) < \frac{C}{2}\}$, there is a joint upper bound

$$\frac{\Phi}{\rho^\alpha (d+u)^\alpha} \le d_3, \qquad \frac{\| \operatorname{grad} f \|^2}{\rho^\alpha (d+u)^\alpha} \le d_3$$

for some constant $d_3 > 0$ that depends only on β and C.

Proposition 3.7. (Proposition 5.6.14 in [L-X-S-J]). *Let $\Omega \subset \mathbb{R}^2$ be a normalized convex domain. Let $f \in \mathcal{S}(\Omega, C)$ be a smooth and strictly convex function defined in Ω, which satisfies the PDE (1.2) with $\beta > 0$. Assume that there are constants $d_3 > 0$ and $\alpha > 1$ such that, in Ω,*

$$\frac{\Phi}{\rho^\alpha (d+u)^\alpha} < d_3, \qquad \frac{1}{\rho(d+u)} < d_3.$$

Then there exists a constant $d_5 > 0$, depending only on α, β, d_3 and C, such that, on Ω:

$$\exp\left\{-\frac{32(2+d_3)C}{C-f}\right\} \frac{\sum f_{ii}}{\rho^\alpha (d+u)^{\alpha+2}} \le d_5 .$$

For affine maximal surfaces we have $\beta = 0$ in the PDE (1.2). We already stated above that, for the equation of affine maximal surfaces, we may choose an appropriate coordinate system reducing our problem to the case that $\|\operatorname{grad} f\|_E$ is bounded above, and then proving that $\frac{1}{\rho} \le b$ for a certain constant b. In this case we have

Proposition 3.8. (Proposition 5.6.13 in [L-X-S-J]). *Let $\Omega \subset \mathbb{R}^2$ be a normalized convex domain and $0 \in \Omega$ be the center of Ω. Let f be a strictly convex C^∞ function defined on Ω. Assume that*

$$\inf_{\Omega} f = 0, \qquad f = C > 0 \quad on \ \partial\Omega,$$

and that f satisfies the PDE (1.2) with $\beta = 0$; moreover assume that there is a constant $b > 0$ such that, in Ω:

$$\frac{1}{\rho} \le b.$$

Then there exists a real constant $\alpha > 0$ such that the following estimates hold on $\Omega_{\frac{C}{2}}$:

$$\frac{\Phi}{\rho^\alpha} \le d_4, \qquad \frac{\| \operatorname{grad} f \|^2}{\rho^\alpha} \le d_4$$

for some real constant $d_4 > 0$ that depends only on α, b and C.

Proposition 3.9. (Proposition 5.6.15 in [L-X-S-J]). *Let $x_3 = f(x_1, x_2)$ be a smooth and strictly convex function defined on a normalized convex domain $\Omega \subset \mathbb{R}^2$, which satisfies the equation (1.2) with $\beta = 0$. Assume that there exist constants $\alpha \ge 0$ and $d_4 \ge 0$ and a function $f \in \mathcal{S}(\Omega, C)$ such that:*

$$\frac{\Phi}{\rho^\alpha} \le d_4, \qquad \frac{1}{\rho} \le d_4$$

on $\bar{\Omega}$. Then there is a real constant $d_5 > 0$, depending only on α, d_4 and C, such that on Ω

$$\exp\left\{-\frac{32(2+d_4)C}{C-f}\right\}\frac{\sum f_{ii}}{\rho^\alpha(d+u)^{\alpha+2}} \leq d_5\,.$$

Remark. It might be helpful for the understanding of the foregoing proofs to realize certain differences in the proofs, namely:
(1) For $\beta > 0$ we have the estimate of the determinant

$$\frac{1}{(d+u)^4}\cdot\det(f_{ij}) \leq b_0 \qquad for \quad x \in S_f(C') = \{x \in \Omega \mid f(x) \leq C'\}$$

from Lemma 3.1, while for $\beta = 0$ we don't have such an estimate. But in the latter case the PDE (1.2) is affinely invariant, thus we can choose affine coordinates x_1, x_2, x_3 such that $x_3 = f(x_1, x_2)$ satisfies $\|grad\ f\|_E \leq C_0$ for some constant $C_0 > 0$; as a consequence, from Lemma 3.2, we then have the estimate

$$\det(f_{ij}) \leq b_0 \quad for \quad x \in S_f(C').$$

(2) The assumptions in Proposition 3.6 and Proposition 3.8 are different, therefore we can not prove Proposition 3.8 as Corollary from Proposition 3.6, just considering the limit $\beta \to 0$.

In [L-X-S-J] and [L-J-1] we proved in detail both Lemmas 5.6.10 and 5.6.11, thus also Proposition 5.6.12 in [L-X-S-J] has been proved in all details. As the reader can realize, the proofs of Proposition 5.6.13 and 5.6.15 are, step by step, similar to the proofs of Propositions 5.6.12 and 5.6.14, including the method, the estimates, and even the details of computation; therefore we omitted the proofs of Propositions 5.6.13 and 5.6.15 in both publications, [L-J-1] and [L-X-S-J].

Second Proof of Lemma 2.1.

This proof is taken from [L-X-S-J], p. 129; it uses pure analysis.
Proof. We treat the two cases $\beta > 0$ and $\beta = 0$ separately.

Case $\beta > 0$. Let $0 \in \Omega_k$ be the center of Ω_k and $u^{(k)}$ the Legendre transformation function of $f^{(k)}$ relative to 0.
To simplify the notations we will use $f^{(k)}$ to denote $f^{(i_\ell)}$. Lemma 3.1 and Propositions 3.6 and 3.7 imply the following uniform estimates:

$$\frac{\Phi^{(k)}}{\rho^\alpha(d+u^{(k)})^\alpha} \leq d_6, \quad \frac{1}{\rho^\alpha(d+u^{(k)})^\alpha} \leq d_6, \quad \frac{\sum f_{ii}^{(k)}}{\rho^\alpha(d+u^{(k)})^{\alpha+2}} \leq d_6$$

in

$$S_{f^{(k)}}(q^k, \tfrac{C}{2}) := \{x \in \Omega_k \mid f^{(k)} < \tfrac{C}{2}\},$$

where d_6 is a positive constant depending only on β and C. We may assume that q_k converges to p_o. Let $B_R(q^k)$ be a Euclidean ball with $\Omega \subset B_{\frac{R}{2}}(q^k)$. Then the Legendre transformation domain of Ω satisfies $B_\delta^*(0) \subset \Omega^*$, where $\delta = \frac{C}{2R}$ and $B_\delta^*(0) = \{\xi \mid \xi_1^2 + \xi_2^2 < \delta^2\}$. Lemma 3.3 gives

$$\det(f_{ij}) \geq b_3$$

for $\xi \in B^*_{\frac{\delta}{2}}(0)$, where b_3 is a constant depending only on C and β. Restricting to $B^*_\delta(0)$, we have

$$-\frac{C}{2R} - C \le u^{(k)} = \sum \xi_i x_i - f^{(k)} \le \frac{C}{2R}.$$

Therefore the sequence $u^{(k)}$ locally uniformly converges to a convex function u^∞ in $B^*_\delta(0)$, and there are constants $0 < \lambda \le \Lambda < \infty$ such that the following estimates hold in $B^*_{\frac{\delta}{2}}(0)$

$$\lambda \le \lambda_i^{(k)} \le \Lambda, \qquad \text{for} \quad i = 1, 2, \cdots, n, \quad k = 1, 2, \cdots$$

where $\lambda_1^{(k)}, \cdots, \lambda_n^{(k)}$ denote the eigenvalues of the matrix $(f_{ij}^{(k)})$. Standard elliptic estimates finally prove the assertion of Lemma 2.1 in the case $\beta > 0$.

Case $\beta = 0$. Denote $D := \{x \mid f(x) = 0\}$. Again we have two subcases.

(i) If $D \bigcap \partial\Omega = \emptyset$ then there is a constant $h > 0$ such that the level set satisfies $\bar{S}_f(p_o, h) \subset \Omega$, thus we have a uniform estimate for $\|grad\, f^{(k)}\|_E$ in $\bar{S}_{f^{(k)}}(q^k, h)$. From Lemma 3.2 it follows that there is a uniform estimate for $\frac{1}{\rho}$ in $\bar{S}_f^{(k)}(q^k, \frac{h}{2})$. Now we use Propositions 3.8 and 3.9 and the same argument as in the case $\beta > 0$ in the second proof to complete the proof.

(ii) In case $D \bigcap \partial\Omega \ne \emptyset$, let $p \in D \bigcap \partial\Omega$. Since the PDE (1.2) with $\beta = 0$ is equiaffinely invariant, we may choose a new coordinate system such that $\|grad\, f^{(k)}\|_E$ is uniformly bounded in $\bar{S}_{f^{(k)}}(p, h)$. Then the same argument as in (i) above shows that f is smooth in a neighborhood of p, and we get a contradiction. This excludes the case $D \bigcap \partial\Omega \ne \emptyset$ and thus completes the proof of Lemma 2.1. ∎

4. Proof of Proposition 3.8

When formulating Proposition 3.8, we already have stated that this Proposition is exactly Proposition 5.6.13 in [L-X-S-J], but its proof was omitted in our monograph for the reasons given above.

To emphasize the similarity of both proofs we exceptionally use a notation that we introduced in section 3.2 above for similar, but not identical expressions; this might be allowed as the proof of both Lemmas, namely 3.4 and 3.5, has been finished. Now we define:

$$\mathcal{A} := \max_\Omega \left\{ \exp\left\{ -\frac{m}{C-f} \right\} \frac{\Phi}{\rho^\alpha} \right\},$$

$$\mathcal{D} := \max_\Omega \left\{ \exp\left\{ -\frac{m}{C-f} + K \right\} \frac{g^2 \, \| grad\, f \|^2}{\rho^\alpha} \right\},$$

where

$$K := \frac{N}{\mathcal{A}} \exp\left\{ -\frac{m}{C-f} \right\} \frac{\Phi}{\rho^\alpha},$$

and m, α and N are positive constants to be determined later.

It is important for the understanding of the following proof to realize that we use two different differential inequalities for the expression $\frac{\Delta\Phi}{\Phi}$, namely one coming

from the affine maximal surface equation, the other from the study of the function L below. The steps of the proof of Proposition 3.8 are the following:

Step 1. First we will prove that $\mathcal{A} \leq \frac{4}{\alpha}\mathcal{D}$. The method is the same as in the proof of Lemma 5.6.10 in [L-X-S-J]. Consider the function

$$L := \exp\left\{-\frac{m}{C-f}\right\} \frac{\Phi}{\rho^\alpha}$$

defined on Ω. From the definition of L and its behavior at the boundary, the function L attains its supremum at some interior point p^*. Differentiation gives at p^*:

$$\frac{\Phi_{,i}}{\Phi} - g f_{,i} - \alpha\frac{\rho_{,i}}{\rho} = 0, \tag{4.1}$$

$$\frac{\Delta\Phi}{\Phi} - \frac{\|\operatorname{grad}\Phi\|^2}{\Phi^2} + \alpha\Phi - g'\|\operatorname{grad} f\|^2 - g\Delta f \leq 0; \tag{4.2}$$

here in (4.2) and later we use the abbreviation

$$g := \frac{m}{(C-f)^2}, \qquad g' := \frac{2m}{(C-f)^3}.$$

We choose $m = 2C$. Then $g' \leq g^2$. Then the inequality in (4.2) gives an upper bound for the expression $\frac{\Delta\Phi}{\Phi}$.

On the other hand, the affine maximal surface equation (see Proposition 4.5.2 in [L-X-S-J], in this case $n = 2$ and $\beta = \delta = 0$) gives the following lower bound for $\frac{\Delta\Phi}{\Phi}$:

$$\Delta\Phi \geq \frac{\|\operatorname{grad}\Phi\|^2}{\Phi} + \Phi^2. \tag{4.3}$$

A combination of the inequalities (4.2), (4.3) and another application of the inequality of Schwarz give

$$\exp\left\{-\frac{m}{C-f}\right\}\frac{\Phi}{\rho^\alpha} \leq \frac{2}{\alpha}\exp\left\{-\frac{m}{C-f}\right\}g^2\frac{\|\operatorname{grad} f\|^2}{\rho^\alpha} + \frac{4b\alpha}{\alpha C}.$$

We may assume that, at p^*, $\exp\left\{-\frac{m}{C-f}\right\}g^2\frac{\|\operatorname{grad} f\|^2}{\rho^\alpha} > \frac{2b\alpha}{C}$. Note that $K(p^*) = N$. Then Step 1 is proved.

Step 2. We will prove that $\mathcal{D}$ has an upper bound. Consider the function (see (5.6.6) in [L-X-S-J])

$$F := \exp\left\{\frac{-m}{C-f} + \tau\right\} Q\,\|\operatorname{grad} h\|^2$$

defined on Ω, and put

$$\tau := K, \quad Q := \frac{g^2}{\rho^\alpha}, \quad h := f.$$

Assume that F attains its supremum at the point q^*. Choose a local orthonormal frame field with respect to the Calabi metric near q^* such that $f_{,1}(q^*) = \|\operatorname{grad} f\|$, and choose $\delta = \frac{1}{10}$, $N \gg 10$. Then a calculation as in the proof of Lemma 5.6.11 in [L-X-S-J] gives, at q^*:

$$\left(-g f_{,i} + \frac{4}{C-f}f_{,i} - \alpha\frac{\rho_{,i}}{\rho} + K_{,i}\right)(f_{,1})^2 + 2\sum f_{,j}f_{,ji} = 0, \tag{4.4}$$

$$2(f_{,11})^2 + 2(f_{,12})^2 + 4\frac{\rho_{,11}}{\rho}(f_{,1})^2 + (\alpha - 388)\Phi(f_{,1})^2 - 328 + \Delta K(f_{,1})^2$$

$$- \left(g'(f_{,1})^2 + 2\left(g - \tfrac{4}{C-f}\right) + 2\left(g - \tfrac{4}{C-f}\right)\tfrac{\rho_{,1}}{\rho}f_{,1}\right)(f_{,1})^2 \le 0. \qquad (4.5)$$

In the following we calculate estimates for three terms appearing in (4.5), namely the terms:

- $(f_{,11})^2 + (f_{,12})^2$.

- ΔK.

- $4\tfrac{\rho_{,11}}{\rho}(f_{,1})^2$, resp.

Again we emphasize that these steps and details of our calculation are similar to the steps and details in case that $\beta > 0$; see pp. 123-126 in [L-X-S-J].

Estimate for the term $(f_{,11})^2 + (f_{,12})^2$.

$$2(f_{,11})^2 = \tfrac{1}{2}\left[gf_{,1} - \tfrac{4}{C-f}f_{,1} + \alpha\tfrac{\rho_{,1}}{\rho} - K_{,1}\right]^2 (f_{,1})^2$$

$$\ge \tfrac{3}{4N}\left[\left(g - \tfrac{4}{C-f}\right)f_{,1} + \alpha\tfrac{\rho_{,1}}{\rho}\right]^2 (f_{,1})^2 - \tfrac{9}{10}\tfrac{(K_{,1})^2}{K}(f_{,1})^2, \qquad (4.6)$$

$$2(f_{,12})^2 \ge \tfrac{3}{4N}\alpha^2(\tfrac{\rho_{,2}}{\rho})^2(f_{,1})^2 - \tfrac{9}{10}\tfrac{(K_{,2})^2}{K}(f_{,1})^2. \qquad (4.7)$$

Estimate for the term ΔK.

$$K_{,i} = K\left(\tfrac{\Phi_{,i}}{\Phi} - \alpha\tfrac{\rho_{,i}}{\rho} - gf_{,i}\right), \qquad (4.8)$$

$$\Delta K \ge \tfrac{\|\text{grad } K\|^2}{K} - 2Kg\tfrac{\rho_{,1}f_{,1}}{\rho} - N\left(g'(f_{,1})^2 + 2g\right). \qquad (4.9)$$

Estimate for the term $4\tfrac{\rho_{,11}}{\rho}(f_{,1})^2$.

Note that $\beta = 0$ now, by the same estimates as in Lemma 5.6.11 ([L-X-S-J], p. 124), we have

$$\sum \tfrac{(\rho_{,ij})^2}{\rho^2} \le \tfrac{2((\rho_{,11})^2+(\rho_{,12})^2)}{\rho^2} \le \tfrac{\sum(\Phi_{,i})^2}{\Phi} + 4\Phi^2.$$

It follows that

$$4\tfrac{|\rho_{,11}|}{\rho}(f_{,1})^2 \le 4\tfrac{\|\text{grad }\Phi\|}{\sqrt{\Phi}}(f_{,1})^2 + 8\,\Phi(f_{,1})^2 \le 8\Phi(f_{,1})^2 +$$

$$+4\sqrt{2}\sqrt{\Phi}\left(\left[\sum\left(\tfrac{\Phi_{,i}}{\Phi} - gf_{,i} - \alpha\tfrac{\rho_{,i}}{\rho}\right)^2\right]^{\frac{1}{2}} + \left[\sum\left(gf_{,i} + \alpha\tfrac{\rho_{,i}}{\rho}\right)^2\right]^{\frac{1}{2}}\right)(f_{,1})^2. \quad (4.10)$$

We apply the inequality of Schwarz and (4.8) to get

$$4\sqrt{2}\sqrt{\Phi}\left[\sum\left(\tfrac{\Phi_{,i}}{\Phi} - gf_{,i} - \alpha\tfrac{\rho_{,i}}{\rho}\right)^2\right]^{\frac{1}{2}}(f_{,1})^2 \le$$

$$\le \tfrac{1}{20}\tfrac{\|\text{grad } K\|^2}{K}(f_{,1})^2 + \tfrac{240}{N}\mathcal{A}\exp\left\{\tfrac{m}{C-f}\right\}\rho^\alpha(f_{,1})^2,$$

$$4\sqrt{2}\sqrt{\Phi}\left[\sum\left(gf_{,i} + \alpha\tfrac{\rho_{,i}}{\rho}\right)^2\right]^{\frac{1}{2}}(f_{,1})^2 \le 300N\Phi(f_{,1})^2 +$$

$$+\tfrac{1}{12N}\sum\left[\left(g-\tfrac{4}{C-f}\right)f_{,i}+\alpha\tfrac{\rho_{,i}}{\rho}\right]^2(f_{,1})^2+\tfrac{4}{3N}\tfrac{1}{(C-f)^2}(f_{,1})^4.$$

Note that $K(p^*)=N$. From Step 1 we get

$$\exp\{N\}\mathcal{A}\le\tfrac{4}{\alpha}\exp\left\{-\tfrac{m}{C-f}+K\right\}\tfrac{g^2\|\operatorname{grad}f\|^2}{\rho^\alpha}(q^*).$$

It follows that, at q^*,

$$\tfrac{240}{N}\mathcal{A}\exp\left\{\tfrac{m}{C-f}\right\}\rho^\alpha(f_{,1})^2\le\tfrac{960}{N\alpha}\,g^2(f_{,1})^4.$$

We insert this estimate into (4.10) to finally get the third estimate:

$$4\tfrac{|\rho_{,11}|}{\rho}(f_{,1})^2\le\tfrac{1}{20}\tfrac{\|\operatorname{grad}K\|^2}{K}(f_{,1})^2+\tfrac{960}{N\alpha}g^2(f_{,1})^4+\tfrac{4}{3N}\tfrac{1}{(C-f)^2}(f_{,1})^4$$

$$+302N\Phi(f_{,1})^2+\tfrac{1}{12N}\sum\left[\left(g-\tfrac{4}{C-f}\right)f_{,i}+\alpha\tfrac{\rho_{,i}}{\rho}\right]^2(f_{,1})^2.\qquad(4.11)$$

After finishing the proof of the three estimates, we use the inequalities (4.6), (4.7) and (4.9) and insert into (4.5); we get

$$\tfrac{2}{3N}\sum\left[\left(g-\tfrac{4}{C-f}\right)f_{,i}+\alpha\tfrac{\rho_{,i}}{\rho}\right]^2(f_{,1})^2+(\alpha-340N)\,\Phi(f_{,1})^2$$

$$-2(N+1)g(f_{,1})^2-2\left(Kg+g-\tfrac{4}{C-f}\right)\tfrac{\rho_{,1}}{\rho}(f_{,1})^3$$

$$-\left[(N+1)g'+\tfrac{1000}{N\alpha}g^2+\tfrac{4}{3N}\tfrac{1}{(C-f)^2}\right](f_{,1})^4-328\le0.\qquad(4.12)$$

As in the proof of Lemma 5.6.11 in [L-X-S-J] we choose N and α such that

$$1+N=\tfrac{2\alpha}{3N},\quad i.e.,\;\;\alpha=\tfrac{3N(1+N)}{2},\qquad(4.13)$$

moreover we choose N large enough that $N>10^6$, and finally we choose $m\ge2C\alpha N(N+1)$; then

$$g'\,(N+1)\le\tfrac{1}{N\alpha}\,g^2,\qquad\tfrac{4}{3N}\,\tfrac{1}{(C-f)^2}<\tfrac{1}{N^2\alpha}\,g^2.$$

Again, as in the proof of Lemma 5.6.11 in [L-X-S-J], we discuss two cases:

Case 1: $\sum\tfrac{\rho_{,i}f_{,i}}{\rho}>0.$ In this case (4.13) gives the inequality:

$$\tfrac{2}{3N}\left[\left(g-\tfrac{4}{C-f}\right)f_{,i}+\alpha\tfrac{\rho_{,i}}{\rho}\right]^2(f_{,1})^2-(2+2N)\left(g-\tfrac{4}{C-f}\right)\tfrac{\rho_{,1}}{\rho}(f_{,1})^3$$

$$\ge\tfrac{2}{3N}\left(g-\tfrac{4}{C-f}\right)^2(f_{,1})^4\ge\tfrac{1}{3N}g^2(f_{,1})^4.$$

Note that $K\le N$, we have

$$2N\left(g-\tfrac{4}{C-f}\right)\tfrac{\rho_{,1}f_{,1}}{\rho}-2gK\tfrac{\rho_{,1}f_{,1}}{\rho}\ge-\tfrac{8N}{C-f}\tfrac{\rho_{,1}f_{,1}}{\rho}\ge-200N\Phi-\tfrac{3}{50\alpha}g^2(f_{,1})^2.$$

Now we insert the two estimates above into (4.12) and get:

$$\tfrac{1}{6N}g^2(f_{,1})^4-2(N+1)g(f_{,1})^2-328\le0.$$

Case 2: $\sum \frac{\rho_{,i} f_{,i}}{\rho} \leq 0.$ The inequality of Schwarz implies

$$\frac{2}{3N} \sum \left[\left(g - \frac{4}{C-f} \right) f_{,i} + \alpha \frac{\rho_{,i}}{\rho} \right]^2 (f_{,1})^2 + \frac{\alpha}{4} \Phi (f_{,1})^2 \geq$$

$$\geq \frac{2}{8\alpha+3N} \left(g - \frac{4}{C-f} \right)^2 (f_{,1})^4 \geq \frac{1}{8\alpha+3N} g^2 (f_{,1})^4.$$

We insert this estimate into (4.12) and obtain the inequality

$$\frac{1}{16\alpha+6N} g^2 (f_{,1})^4 - 2(N+1) g (f_{,1})^2 - 328 \leq 0.$$

Thus, in both cases (1) and (2), we have an inequality of the type

$$a_0 g^2 (f_{,1})^4 - a_1 g (f_{,1})^2 - 328 \leq 0,$$

where a_0 and a_1 are real positive constants. Consequently

$$\exp \left\{ -\frac{m}{C-f} + K \right\} g^2 \frac{1}{\rho^\alpha} \| \operatorname{grad} f \|^2 \leq a_2,$$

where a_2 is a real positive constant depending only on C, α and b; this is the upper bound for $\mathcal{D}$ that we announced in the beginning of Step 2.

The upper bound for $\mathcal{D}$ gives an upper bound for $\mathcal{A}$ (see Step 1), thus the proof of Proposition 3.8 (Proposition 5.6.13 in [L-X-S-J]) is finished.

In [L-X-S-J], p. 126, we stated that the proof of Proposition 5.6.13 there (that is Proposition 3.8 here) is "similar" to that of Proposition 5.6.12 in the same monograph (that is Proposition 3.6 here). We hope that our foregoing proof, containing all details of the auxiliary tools, will convince the reader that this statement of "similarity" is correct.

This finishes this proof of Chern's Affine Bernstein Conjecture.

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